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Covariant Sectors with Infinite Dimension and Positivity of the Energy

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Abstract

Let $\mathcal{A}$ be a local conformal net of von Neumann algebras on S^1 and ρ a Möbius covariant representation of $\mathcal{A}$, possibly with infinite dimension. If ρ has finite index, ρ has automatically positive energy. If ρ has infinite index, we show the spectrum of the energy always to contain the positive real line, but, as seen by an example, it may contain negative values. We then consider nets with Haag duality on $\mathbb{R}$, or equivalently sectors with non-solitonic extension to the dual net; we give a criterion for irreducible sectors to have positive energy, namely this is the case iff there exists an unbounded Möbius covariant left inverse. As a consequence the class of sectors with positive energy is stable under composition, conjugation and direct integral decomposition.

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1 Introduction

In the past there has been a certain belief that all irreducible superselection sectors have finite statistics. Indeed if $\mathcal{A}$ is a translation covariant net on the Minkowski spacetime and the energy-momentum spectrum has an isolated mass shell, then, by a theorem of Buchholz and Fredenhagen [8], every positive-energy irreducible representation is localizable in a space-like cone and have finite statistics.

Nevertheless an analysis of sectors with infinite dimension is possible, in the context of modular covariant net, as outlined in [18], and this indicated such quantum charges to have a natural occurrence. First examples of irreducible superselection sectors with infinite dimension have been constructed, in a simple way, by Fredenhagen [14], associated with conformal nets on S^1, and moreover there are arguments that in this context a large natural family of sectors should have infinite dimension [22], thus providing the feeling that infinite statistics might be be the generic or prevailing situation in low spacetime dimension.

At this point it is natural to begin with a general study of superselection sectors with infinite dimension. However the extension from the finite-dimensional to the infinite dimensional case is certainly far from being straightforward and requires new methods and insight; it is analogous to the passage, in the study of group representations, from compact groups to locally compact groups.

In order to understand the structure of infinite dimensional sectors, we shall study here the positive energy property. We start with a study of the finite index case in the one-dimensional conformal case, with a point of view suitable for generalization. Classical arguments, see [11], are replaced also due to the failure of Haag duality on the real line and the occurrence of soliton sectors. Based on modular theory methods, we show however that the positivity of the energy holds automatically in the finite index case.

In the context of infinite dimensional sectors we shall then show that the spectrum of the energy always contains $\mathbb{R}_+$. But, as we shall illustrate by a (reducible) example, negative energy values may occur in general.

We are thus led to characterize the sectors with positive energy. To this end we study the basic question whether we may associate an unbounded left inverse with every covariant superselection sector with positive energy. We start by considering a translation-dilation covariant net $\mathcal{A}$ of von Neumann algebras on $\mathbb{R}$ obtained by a Möbius covariant precosheaf on S^1 by cutting the circle at one point. Such translation-dilation covariant nets are characterized by the Bisognano-Wichmann geometric action of the modular group of the von Neumann algebras of half-lines, see [19].

Assuming Haag duality for $\mathcal{A}$ on $\mathbb{R}$ to hold (for bounded intervals), we give a positive answer to the above question and in fact we show that an irreducible sector has positive energy if and only if it admits a Möbius covariant unbounded left inverse.

As a consequence we show the class of superselection sectors with positive energy is closed under composition, conjugation and direct integral decomposition.

Notice now that the Bisognano-Wichmann dual net $\mathcal{A}^d$ of $\mathcal{A}$

$$\mathcal{A}^d(a,b) = \mathcal{A}(-\infty,b) \cap \mathcal{A}(a,+\infty)$$

always satisfies Haag duality, is conformal thus strongly additive [19], and a covariant representation ρ of $\mathcal{A}$ localized in a bounded interval I extends to a covariant representation of $\mathcal{A}^d$, but it may become localized in a half-line, i.e. a soliton sector [24]. However one may construct an extension ρ_R of ρ to $\mathcal{A}^d$ localized in a right half-line and an extension ρ_L localized in a left half-line. The extensions are easily obtained: if $I \subset (a,+\infty)$, the restriction of ρ to the C*-algebra generated by the von Neumann algebras of bounded intervals contained in $(a,+\infty)$ extends to a normal endomorphism of the its weak closure $\mathcal{A}(a,+\infty)$, thus, as $a \in \mathbb{R}$ is arbitrary, it gives up an endomorphism of the C*-algebra $\cup_{a\in\mathbb{R}}\mathcal{A}(a,+\infty)^-$ that restricts to a representation of the quasi-local C*-algebra of $\mathcal{A}^d$ generated by the local von Neumann algebras $\mathcal{A}^d(I)$'s, I bounded interval. This representation is ρ_R and ρ_L is obtained similarly. In general ρ_R and ρ_L are inequivalent representations.

They are equivalent in particular if they are still localized in a bounded interval, namely the extension of ρ is not a soliton. Our results thus apply to general (non necessarily strongly additive) Möbius covariant nets, provided we consider 'truly non-solitonic' covariant sectors.

The idea behind our analysis is to explore the equivalence between the positivity of the energy and the KMS condition for the dilation automorphisms of $\mathcal{A}(\mathbb{R}_+)$, that holds in the vacuum sector. It is rather easy that the latter KMS property passes to charged sectors, of arbitrary dimension, enabling us to make an analysis by the Tomita–Takesaki theory. Our methods rely on an analysis of the unitary representation of translation-dilation group, where we give a characterization of the positive energy representations in terms of domain conditions for certain associated operators. We then identify these operators with modular objects furnished by the Tomita-Takesaki theory and use the domain conditions. In the infinite index case we make use of Haagerup operator–valued weights, Connes spatial derivatives and Araki modular operators in particular. For the convenience of the reader we use [25] as a reference for the modular theory.

This paper leaves open the problem whether every irreducible covariant sector has automatically positive energy. Our work however indicates that, at least in the strongly additive case, the answer is likely to be affirmative.

2 Representations of the translation-dilation group: a criterion for positive energy

In this section we analyze the unitary positive-energy representations of the dilation-translation group from a point of view of later use. In particular we are interested in characterizing the positive energy representations in terms of domain conditions. To begin with, we relax the hypothesis of positivity of the energy in the proof of [21], Cor. 2.8.

2.1 Lemma. *Let $T(a) := e^{iHa}$, $U(t)$ be two 1–parameter groups on a Hilbert space $\mathcal{H}$ such that $U(t)T(a)U(t)^* = T(e^{-2\pi t}a)$ for every $t, a \in \mathbb{R}$. Then the spectral projection P_1 (resp. P_2, P_3), relative to the positive (resp. negative, 0) part of the spectrum of H commutes with T, U, and thus reduces the representation on globally invariant subspaces.*

Proof. From the commutation relation $U(t)T(a)U(t)^* = T(e^{-2\pi t}a)$, by differentiation with respect to a, it follows that $U(t)HU(t)^* = e^{-2\pi t}H$. For $i = 1, 2, 3$ let χ_i be the characteristic function of the positive (resp. negative, zero) part of $\mathbb{R}$. By Borel functional calculus we get: $\chi_i(U(t)HU(t)^*) = U(t)\chi_i(H)U(t)^* = \chi_i(e^{-2\pi t}H) = \chi_i(H)$. As $\chi_i(H) = P_i$, we have the proof. $\square$

2.2 Lemma. *Given two 1-parameter unitary groups $T(a)$, $U(t) = e^{-2\pi itD}$ on the Hilbert space $\mathcal{H}$ such that $U(t)T(a)U(t)^* = T(e^{-2\pi t}a)$ for every $t, a \in \mathbb{R}$, then for every $a \in \mathbb{R}$ and ζ in the domain of both $e^{-\pi D}$, $e^{-\pi D}T(a)$, we have*

$$\|e^{-\pi D}\zeta\| = \|e^{-\pi D}T(a)\zeta\|.$$

The subspace $\mathcal{D}(e^{-\pi D}) \cap \mathcal{D}(e^{-\pi D}T(a))$ of such ζ is dense for every fixed $a \in \mathbb{R}$.

Proof. By symmetry considerations it is sufficient to consider the case $a \geq 0$. If the generator $-i\frac{d}{da}T(a)|_{a=0}$ of T is non–negative then $\mathcal{D}(e^{-\pi D}) \subset \mathcal{D}(e^{-\pi D}T(a))$, $a \geq 0$ and the thesis is well known [21] (see also Prop. 3.4). We just outline how to change the arguments in this reference in order to have our general statement.

Let us decompose the Hilbert space $\mathcal{H}$ in the direct sum of the two spectral subspaces $\mathcal{H}_+ := (P_1 + P_3)\mathcal{H}$, $\mathcal{H}_- := P_2\mathcal{H}$. By Lemma 2.1 these subspaces are invariant for both the 1-parameter groups $T(a)$ and $U(t)$, so that it is possible to define the restrictions $U_+(t)$, (resp. $U_-(t)$) and $T_+(a)$, (resp. $T_-(a)$) of the 1-parameter groups to these subspaces. Now $T_+(a)$ (resp. $T_-^{-1}(a)$) is a 1-parameter group with positive generator satisfying the commutation relation $U_+(t)T_+(a)U_+(t)^* = T_+(e^{-2\pi t}a)$ (resp. $U_-(t)T_-^{-1}(a)U_-(t)^* = T_-^{-1}(e^{-2\pi t}a)$,) therefore using [21], Corollary 2.8, we obtain: $\|e^{-\pi D_+}\zeta\| = \|e^{-\pi D_+}T_+(a)\zeta\|$ (resp. $\|e^{-\pi D_-}\zeta\| = \|e^{-\pi D_-}T_-^{-1}(a)\zeta\|$) for every ζ in the domain of $e^{-\pi D_+}$, (resp. for every ζ in the domain of $e^{-\pi D_-}$), for every $a \geq 0$, where D_+ and D_- are the generators of the 1-parameter groups $U_+(t)$, $U_-(t)$. Let us take $\zeta = (\zeta_+ + \zeta_-)$ in the common domain of $e^{-\pi D}$ and $e^{-\pi D}T(a)$, $a \geq 0$, then we have: ζ_+ in the domain of $e^{-\pi D_+}$ and ζ_- in the domain of $e^{-\pi D_-}$ so that we obtain $\|e^{-\pi D_+}\zeta_+\| = \|e^{-\pi D_+}T_+(a)\zeta_+\|$ and $\|e^{-\pi D_-}\zeta_-\| = \|e^{-\pi D_-}T_-^{-1}(a)\zeta_-\|$. From the second equation (using the fact that by hypothesis $T_-(a)\zeta_-$ is in the domain of $e^{-\pi D_-}$), we get $\|e^{-\pi D_-}T_-(a)\zeta_-\| = \|e^{-\pi D_-}T_-^{-1}(a)T_-(a)\zeta_-\| = \|e^{-\pi D_-}\zeta_-\|$. Now, summing the result for the two components ζ_+, ζ_-, from Pitagora's Theorem it is possible to deduce: $\|e^{-\pi D}\zeta\| = \|e^{-\pi D}T(a)\zeta\|$, $a \geq 0$. If $a > 0$ and $\zeta \in \mathcal{D}(e^{-\pi D}) \subset \mathcal{D}(e^{-\pi D}T(-a))$, then $T(-a)\zeta \in \mathcal{D}(e^{-\pi D}) \subset \mathcal{D}(e^{-\pi D}T(a))$, thus $\|e^{-\pi D}\zeta\| = \|e^{-\pi D}T(a)T(-a)\zeta\| = \|e^{-\pi D}T(-a)\zeta\|$. The last part of the statement is now clear. $\square$

2.3 Corollary. *Assume that $T(a)$, and $U(t) = e^{-2\pi itD}$ are two 1–parameter groups as in Lemma 2.2, and let $\zeta \in \mathcal{H}$ be a vector such that $\|e^{-\pi D}\zeta\| =$*

$\|e^{-\pi D}T(a)\zeta\| < \infty$, *for some* $a \geq 0$; *then* $\|e^{-\pi D}\zeta\| = \|e^{-\pi D}T(b)\zeta\|$ *for every* $0 < b < a$.

Proof. With the same notation as above we know that $\zeta_+ \in \mathcal{D}(e^{-\pi D_+})$ and $T_-(a)\zeta_- \in \mathcal{D}(e^{-\pi D_-})$, thus $\zeta_+ \in \mathcal{D}(e^{-\pi D_+}T(b))$ and $T_-(b)\zeta_- = T_-(a-b)^* T_-(a)\zeta_- \in \mathcal{D}(e^{-\pi D_-})$. □

We also have the following criterion for the positivity of the energy

2.4 Proposition. *Given two* 1*-parameter groups* $T(a)$, $U(t) = e^{-2\pi i t D}$ *as in Lemma 2.2, the following are equivalent:*

a) $e^{-\pi D}T(a) \supset T(-a)e^{-\pi D}$, *i.e.* $T(-a)e^{-\pi D}$ *is hermitean, for some (*$\Leftrightarrow$ *for all)* $a > 0$*;*

b) *there exists a core* $\mathcal{D}_1$ *for* $e^{-\pi D}$ *contained in* $\mathcal{D}(e^{-\pi D}T(a))$, *for some (*$\Leftrightarrow$ *for all)* $a > 0$*;*

c) *the generator* H *of* $T(a)$ *is positive.*

Proof. The equivalence a) $\Leftrightarrow$ c) is proved in [10], Th. 1. The implication c) $\Rightarrow$ a) is also contained in [21], (proof of) Cor. 2.8 by a different method of proof. Clearly a) $\Rightarrow$ b), thus we have to show that b) $\Rightarrow$ c). $\mathcal{D}_2 := e^{-\pi D}\mathcal{D}_1$ is a core for $e^{\pi D}$, and, for some $a > 0$, $e^{-\pi D}T(a)e^{\pi D}$ is isometric on $\mathcal{D}_2$ by Lemma 2.2, thus by [26], 9.24 the function $t \to U(t)T(a)U(t)^*$ admits an analytic continuation inside the strip $\{t \in \mathbb{C} \mid -\frac{1}{2} < \Im t < 0\}$ bounded in norm by 1, see also Th. 4.3. The conclusion may be easily obtained as in [5], (proof of) Prop. 2.7; in fact putting $t = -\frac{i}{4}$ we get $\|T(ia)\| = e^{-aH} \leq 1$. □

3 Preliminaries on local conformal precosheaves

Our analysis will concern nets of von Neumann algebras on the real line. More precisely $\mathcal{A}$ will be a map $I \mapsto \mathcal{A}(I)$ from the bounded open intervals I of $\mathbb{R}$ to von Neumann algebras on a fixed Hilbert space $\mathcal{H}$. For this net we require the following properties:

1) **Isotony**: $I_1 \subset I_2 \Rightarrow \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$

2) **Locality**: $I_1 \cap I_2 = \emptyset \Rightarrow [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$;

3) **Covariance**: there exists a strongly continuous unitary representation V of the translation-dilation group P, namely a semidirect product of $\mathbb{R}$ with $\mathbb{R}$, on $\mathcal{H}$ such that:

$$V(g)\mathcal{A}(I)V(g)^{-1} = \mathcal{A}(gI),\ g \in P.$$

Here P acts on $\mathbb{R}$ ($(a,t)x = a + e^t x$) and we will denote by $a, b, \cdots \in \mathbb{R}$ elements of the translation one–parameter subgroup and by $t, s, \cdots \in \mathbb{R}$

elements of dilation one-parameter subgroup. We shall frequently denote the one-parameter translation (resp. dilation) group simply by $T(a)$ (resp. $U(t)$).

4) **Existence of the vacuum**: there exists a unique (up to a phase) unit V-invariant vector $\Omega \in \mathcal{H}$.

5) **Reeh–Schlieder Property**: the vacuum vector Ω is cyclic and separating for the von Neumann algebras $\mathcal{A}(I)$.

6) **Bisognano–Wichmann Property**: the modular unitary one–parameter group associated (by Reeh–Schlieder Property and Tomita–Takesaki Theorem) with $(\mathcal{A}(\mathbb{R}_+), \Omega)$ coincides with the rescaled dilation one–parameter unitary
$$\Delta_\Omega^{it} = U(-2\pi t),\ t \in \mathbb{R}.$$

Here and in the following, given $S \subset \mathbb{R}$, we indicate by $\mathcal{A}_0(S)$ the C^*–algebra generated by all the $\mathcal{A}(I)$, with $I \subset S$, and by $\mathcal{A}(S) = \mathcal{A}_0(S)''$ its weak closure.

In the literature one considers more often Möbius covariant precosheaves (also named nets) of von Neumann algebras on the proper intervals of S^1. If one cuts S^1 and restricts such a precosheaf to $\mathbb{R} = S^1 \backslash \{\text{point}\}$ one obtains a net on the real line satisfying the above properties 1 to 6 [5, 2]. Conversely any net on $\mathbb{R}$ with the above properties extends uniquely to a Möbius covariant precosheaf on S^1 [19].

In particular the modular conjugation $J_{\mathbb{R}}$ associated with $\mathcal{A}(\mathbb{R}_+), \Omega$ corresponds to the reflection in $\mathbb{R}$ with respect to 0, thus "wedge duality" holds:
$$\mathcal{A}(a, \infty)' = \mathcal{A}(-\infty, a).$$
Moreover the generator $H := -i\frac{d}{da}T(a)|_{a=0}$ of the translation one–parameter group T is positive [27, 6]. In other words positivity of the energy in the vacuum sector is a consequence of the KMS property for the dilation automorphism group of $\mathcal{A}(\mathbb{R}_+)$. We will see how this implication works in different representations.

Notice that the uniqueness of the vacuum and the positivity of the energy entail the factoriality of the von Neumann algebras $\mathcal{A}(I)$, if I is a half-line, thus the irreducibility of the quasilocal C*-algebra $\mathcal{A}_0(\mathbb{R})$; indeed every net satisfying the above properties decomposes uniquely into a direct sum of irreducible nets and irreducibility, uniqueness of the vacuum and factoriality of the von Neumann algebras of half-lines are equivalent properties.

We shall now consider a morphism ρ of the quasi-local C*-algebra $\mathcal{A} = \mathcal{A}_0(\mathbb{R})$ localized in a half-line, namely ρ is a representation of $\mathcal{A}$ on $\mathcal{H}$ such that $\rho(X) = X$ for every $X \in \mathcal{A}_0(-\infty, a)$. Two such morphisms ρ, ρ' are said to be equivalent if they are equivalent as representations; thus, by wedge duality, there exists a unitary $T \in \mathcal{A}(a, \infty)$ such that $T\rho(X) = \rho'(X)T$ for every $X \in \mathcal{A}$.

An endomorphism ρ is covariant if there exists a unitary strongly continuous representation $V_\rho : P \to B(\mathcal{H})$ such that:
$$\rho(\alpha_g(X)) = V_\rho(g)\rho(X)V_\rho(g)^{-1},\ X \in \mathcal{A},\ g \in P,$$

where $\alpha_g := \mathrm{Ad}(V(g))$. As far as we consider a covariant irreducible morphism ρ or, more generally, a finite direct sum of irreducibles (in particular finite index endomorphisms), the representation V_ρ providing the covariance is unique, due to the fact that there are no non–trivial finite–dimensional representations of P. In the reducible case, different representations are related by a cocycle in $\rho(\mathcal{A})'$. We shall say that ρ has positive energy if we can choose V_ρ so that the generator of the translation group is positive.

We shall only consider covariant morphism ρ which are *transportable*, i.e. localizable in any half-line $(-\infty, a)$ or (b, ∞). This is in particular the case of a covariant morphism localizable in an interval. Thus ρ is normal on $\mathcal{A}(a,\infty)$ and, if localized in (a,∞), extends to a normal endomorphism of $\mathcal{A}(a,\infty)$ denoted $\rho_{(a,\infty)}$ or simply by ρ, if no confusion arises.

By introducing the notations:

$$\beta^\rho_g := \mathrm{Ad}(V_\rho(g)),\ z_\rho(g) := V_\rho(g)V(g)^*, \quad g \in P,$$

the covariance condition takes the form:

$$\alpha_g \rho \alpha_{g^{-1}} = \mathrm{Ad}(z_\rho(g)^*) \circ \rho \simeq \rho,\ g \in P \tag{3.1}$$

and $z_\rho(g)$ satisfies the following α–cocycle identity:

$$z_\rho(gg') = z_\rho(g)\alpha_g(z_\rho(g')), \quad g, g' \in P.$$

Let ρ be localized in the half-line $I = (a, \infty)$ then $\alpha_g \rho \alpha_{g^{-1}}$ is localized in gI and from formula (3.1) we obtain $z_\rho(g) \in \mathcal{A}((\tilde{I})')' = \mathcal{A}(\tilde{I})$ where $\tilde{I}$ is the largest half-line between I and gI.

We will often use the notations:

$$M := \mathcal{A}(0, +\infty),\ M_b := \alpha_b(M) = \mathcal{A}(b, +\infty),\ M^\rho_b := \beta^\rho_b(M),\ b \in \mathbb{R}_+.$$

As $z_\rho(b) \in M_a$, $b > 0$ it follows that

$$M_b = M^\rho_b, \quad b < a.$$

The Bisognano–Wichmann Property states that the one–parameter group $t \mapsto \alpha_{-2\pi t}$, $t \in \mathbb{R}$, coincides on M with the modular group $t \mapsto \sigma_t := \mathrm{Ad}(\Delta^{it}_\Omega)$, $t \in \mathbb{R}$ and, since the cocycle $z_\rho(-2\pi t)$ is localized in M, by Connes' Theorem there exists a unique semifinite normal faithful (s.n.f.) weight ψ_ρ on M whose Radon–Nikodym derivative with respect to the vacuum state $\omega := (\Omega, \cdot\Omega)$ is given by $(D\psi_\rho : D\omega)_t = z_\rho(-2\pi t)$, see [25], Sect. 11. Then $t \mapsto \beta^\rho_{-2\pi t} = \mathrm{Ad}(z_\rho(-2\pi t)) \circ \alpha_{-2\pi t}$, $t \in \mathbb{R}$, is the modular group associated to the weight ψ_ρ on M.

4 Automatic positivity of the energy in the finite index case

Although we shall be mainly interested in sectors with infinite dimension, our proof will be more transparent by a previous analysis of the finite index case (recall that the index is the square of the dimension, see [20]).

Let ρ be an endomorphism of $\mathcal{A}$ localized in $I \subset \mathbb{R}_+$. In the finite index case the following analog of the Kac-Wakimoto formula holds [21]:

$$(D\varphi_\rho : D\omega)_t = d(\rho)^{-it} z_\rho(-2\pi t),$$

where $\omega = (\Omega, \cdot\Omega)$ is the restriction of the vacuum state to $M = \mathcal{A}(\mathbb{R}_+)$, φ_ρ is the state $\omega \circ \phi_\rho$ on M and $\phi_\rho = \rho^{-1} \circ E_\rho$ is the minimal left inverse of $\rho : M \to M$, with $E_\rho : M \to \rho(M)$ is the minimal conditional expectation.

We recall the following:

4.1 Proposition. [21] *Let us assume that ρ is covariant with finite index as above, and let ψ_ρ be a positive linear functional on M; then the following are equivalent:*

a) *ψ_ρ is normal, faithful and $(D\psi_\rho : D\omega)_t = z_\rho(-2\pi t)$, $t \in \mathbb{R}$*

b) $\psi_\rho = d(\rho)\ \omega \circ \phi_\rho$,

c) $\psi_\rho(XY^*) = (e^{-\pi K_\rho} X\Omega, e^{-\pi K_\rho} Y\Omega)$, $X, Y \in M$,

d) *ψ_ρ is normal faithful, $\sigma_t^{\psi_\rho} \circ \rho = \rho \circ \sigma_t^\omega$, $t \in \mathbb{R}$, and $\psi_\rho|\rho(M)' \cap M$ is a trace whose value on a central projection p is $\psi_\rho(p) = d(\rho_p)$, where ρ_p is the subrepresentation associated to p. (In particular, if ρ is irreducible, this last condition reduces to $\psi_\rho(I) = d(\rho)$).*

Proof. We sketch the first part of the proof. We assume ρ to be irreducible.

We consider the states ω, $\varphi_\rho := \omega \circ \phi_\rho$ on the von Neumann algebra M. Note that $\varphi_\rho \circ \rho = \omega$ i.e. $\varphi_\rho|_{\rho(M)} = \omega \circ \rho^{-1}$. From $\varphi_\rho = \varphi_\rho \circ E_\rho$ it follows that $\rho(M)$ is σ^{φ_ρ}–stable by Takesaki's theorem, therefore

$$\sigma_t^{\varphi_\rho}|_{\rho(M)} = \sigma_t^{\varphi_\rho|_{\rho(M)}} = \rho \circ \sigma_t^\omega \circ \rho^{-1}.$$

Now defining $v_t := (D\varphi_\rho : D\omega)_t \in M$ then we have $v_t \sigma_t^\omega(X) v_t^* = \sigma_t^{\varphi_\rho}(X), X \in M$, thus $v_t \sigma_t^\omega(\rho(X)) v_t^* = \sigma_t^{\varphi_\rho}(\rho(X)) = \rho \circ \sigma_t^\omega \circ \rho^{-1} \rho(X) = \rho \sigma_t^\omega(X) = \beta^\rho_{-2\pi t} \rho(X) = z_\rho(-2\pi t) \sigma_t^\omega(\rho(X)) z_\rho(-2\pi t)^*, X \in M$. Hence

$$z_\rho(-2\pi t)^* v_t \in \sigma_t^\omega(\rho(M))' \cap M = \sigma_t^\omega(\rho(M)' \cap M) = \mathbb{C}.$$

Now to complete the argument as regard to the phase $d(\rho)$ in b), we refer to [21] part 1. The proof of the point c) will be easily obtained by polarization of the first formula contained in Proposition 4.2.

Assuming d), to obtain a), notice that this condition determines $(D\psi_\rho : D\omega)_t$ up to the multiplication by a cocycle in $\rho(M)' \cap M$ hence ψ_ρ is determined by the specification of $\psi_\rho|\rho(M)' \cap M$ that we require to be $d(\rho)$–times the restriction to $\rho(M)' \cap M$ of the minimal expectation of M onto $\rho(M)$. □

Now we have $\beta^\rho_{-2\pi t} = \sigma_t^{\psi_\rho} (= \sigma^{\varphi_\rho}) = \mathrm{Ad}(\Delta_\xi^{it}) = \mathrm{Ad}(\Delta_{\xi,\Omega}^{it})$ on M, where $\varphi_\rho = (\xi, \cdot\xi)$, $\|\xi\| = 1$ and ξ is cyclic for M (e.g. ξ is the vector representative of φ_ρ in the natural cone of M given by Ω). Clearly $\psi_\rho \circ \beta_t^\rho = \psi_\rho$.

Using the fact that ρ is localized in $\mathbb{R}_+$ and recalling the definition of ψ_ρ it is easy to check that $U_\rho(-2\pi t) = z_\rho(-2\pi t)U(-2\pi t) = z_\rho(-2\pi t)\Delta_\Omega^{it}$ coincides up to the phase $d(\rho)^{it}$ with $\Delta^{it}_{\xi,\Omega} = \Delta^{it}_{\varphi_\rho,\omega}$, where $\Delta^{it}_{\xi,\Omega}$ is the Araki relative modular operator, see [4], namely $S_{\xi,\Omega} = J_{\xi,\Omega}\Delta^{\frac{1}{2}}_{\xi,\Omega}$ is the polar decomposition of the closure of $X\Omega \to X^*\xi, X \in M$. In fact we have

$$2\pi K_\rho = -\log\Delta_{\xi,\Omega} - \log d(\rho),$$

see [21]. Hence by the commutation relations of the group P we obtain

$$\Delta^{it}_{\xi,\Omega}T_\rho(a)\Delta^{-it}_{\xi,\Omega} = T_\rho(e^{-2\pi t}a),\ t, a \in \mathbb{R}.$$

The argument given in Lemma 2.2 amounts to a proof that the invariance condition holds if the dimension of ρ is finite, cf. [21], Prop. 2.11.

4.2 Proposition. *Let ρ be a covariant endomorphism with finite dimension localized in $I \subset \mathbb{R}_+$, and let M, $U_\rho(a)$, $a \in \mathbb{R}$, φ_ρ be as above. Then φ_ρ is β^ρ_a–invariant on M, for every $a \geq 0$.*

Proof. If $X \in M$ we have

$$\begin{aligned}\varphi_\rho(X^*X) &= (\xi, X^*X\xi) = \|X\xi\|^2 = \|S_{\xi,\Omega}X^*\Omega\|^2\\ &= \|J_{\xi,\Omega}\Delta^{\frac{1}{2}}_{\xi,\Omega}X^*\Omega\|^2 = \|\Delta^{\frac{1}{2}}_{\xi,\Omega}X^*\Omega\|^2\end{aligned}$$

(see formula (3.3) in [21]), therefore if $a > 0$ and $z_\rho(a)$ denotes the cocycle with respect to the translation by a,

$$\begin{aligned}\varphi_\rho(T_\rho(a)X^*XT_\rho(a)^*) &= \|\Delta^{\frac{1}{2}}_{\xi,\Omega}T_\rho(a)X^*T_\rho(a)^*\Omega\|^2\\ &= \|\Delta^{\frac{1}{2}}_{\xi,\Omega}T_\rho(a)X^*z_\rho(-a)T(a)^*\Omega\|^2\\ &= \|\Delta^{\frac{1}{2}}_{\xi,\Omega}T_\rho(a)X^*z_\rho(-a)\Omega\|^2\\ &= \|T_\rho(a)\Delta^{\frac{1}{2}}_{\xi,\Omega}X^*z_\rho(-a)\Omega\|^2 \ \text{ (by Lemma 2.2)}\\ &= \|\Delta^{\frac{1}{2}}_{\xi,\Omega}(X^*z_\rho(-a))\Omega\|^2 = \|z_\rho(-a)^*X\xi\|^2 = (X\xi, X\xi).\end{aligned}$$

□

In the sequel we write $T\ \eta\ M$ to denote that the (generally unbounded) linear operator T on $\mathcal{H}$ is affiliated to $M \subset B(\mathcal{H})$, see [26], 9.7.

4.3 Proposition. *Let ρ be as above. For any given $a \geq 0$, the function $t \mapsto \Delta^{it}_{\xi,\Omega}T_\rho(a)\Delta^{-it}_{\xi,\Omega} = T_\rho(e^{-2\pi t}a)$ admits an analytic continuation inside the strip $\{z \in \mathbb{C} \mid -\frac{1}{2} < \Im z < 0\}$ which is bounded in norm by* 1.

Proof. The existence of the analytic continuation inside the strip follows from general arguments, see e.g. [7] p. 241. The bound 1 is thus a consequence of Hadamard three line theorem, once we check that the norm of the function is

bounded by 1 on the lines $z = 0$ (this is obvious), $z = -\frac{i}{2} + t, t \in \mathbb{R}$ and has a priori global bound on the entire strip.

We now check that $\Delta_{\xi,\Omega}^{\frac{1}{2}} T_\rho(a) \Delta_{\xi,\Omega}^{-\frac{1}{2}}$, or eqivalently $S_{\xi,\Omega} T_\rho(a) S_{\xi,\Omega}^{-1}$, is extended by an isometric operator. We write for short $S = S_{\xi,\Omega}$.

Note that $S\, T_\rho(a)\, S^{-1}$ is isometric on the dense subspace $M\xi$: given $X \in M$, we have

$$\begin{aligned} S\ T_\rho(a)\ S^{-1}\ X^*\xi = S T_\rho(a) X\Omega = S\ z_\rho(a) T(a) X T(a)^*\Omega \\ = T(a) X^* T(a)^* z_\rho(a)^*\xi = T(a) X^* T_\rho(a)^*\xi \end{aligned}$$

therefore $M\xi \subset \mathcal{D}(ST_\rho(a)S^{-1}) \subset \mathcal{D}(S^{-1})$ and

$$\|\ S\ T_\rho(a)\ S^{-1}\ X^*\xi\| = \|X^*\xi\|,\ X \in M$$

by the β_a^ρ–invariance of φ showed in Prop. 4.2.

As is known $\mathcal{D}(S) = \{T\Omega \mid T\ \eta\ M,\ \Omega \in \mathcal{D}(T),\ \xi \in \mathcal{D}(T^*)\}$, and using Lemma 2.2 it is direct to verify that $\|\ S\ T_\rho(a)\ S^{-1}\ \zeta\| = \|\zeta\|$ for every $\zeta \in \mathcal{D}(S^{-1})$ for which the l.h.s. is well defined.

It remains to check the a priori bound on the strip. This may be derived by the bound on matrix coefficients $|(\Delta_{\xi,\Omega}^{iz} T_\rho(a) \Delta_{\xi,\Omega}^{-iz}\zeta_1, \zeta_2)| \le \|\zeta_1\|\|\zeta_2\|$, with $-\frac{1}{2} < \Im z < 0$, and ζ_1, ζ_2 in spectral subspaces for $\log \Delta_{\xi,\Omega}$ with respect to bounded intervals. The bound follows by the three line theorem, because the involved fuctions are bounded on the strip. □

Notice that the same result may be obtained using the Prop. 2.4 or the Proposition stated in [26], p. 219.

4.4 Corollary. *Let ρ be a covariant endomorphism with finite dimension localized in $I \subset \mathbb{R}_+$, and M, $T_\rho(a)$ as above. Then in the sector ρ the energy (the generator of the 1–parameter group $T_\rho(a)$) is positive.*

Proof. It is immediate from (the proofs of) Prop. 2.4 (the core is $M\Omega$) and Prop. 4.2, cf. Prop. 4.3. □

5 Infinite index (weight) case

5.1 General considerations

We shall now begin an analysis of sectors with infinite dimension, by an extension of the previous methods.

In the following $\mathcal{A}$ is a net on $\mathbb{R}$ as in the previous section, namely $\mathcal{A}$ is obtained by restricting a local conformal precosheaf of von Neumann algebras on S^1.

Let ρ be a covariant endomorphism localized in a half-line strictly contained in $\mathbb{R}_+$ as before, but not necessarily with finite index. We know from [21], Sect. 2, that

$$U_\rho(-2\pi t) =: e^{-2\pi i t K_\rho} = \Delta(\psi_\rho/\omega')^{it} \tag{5.1}$$

(cf. the paragraphs following Prop. 4.1 and Lemma A.2) where ω' is the restriction of the vacuum state to M', and $\Delta(\psi_\rho/\omega')$ is the Connes' spatial derivative with ψ_ρ the weight on M defined at the end of section 3.

Although ψ_ρ is unbounded in general, ω' is a state, represented by the vector Ω. This suggests the formula

$$\psi_\rho(XX^*) = \|e^{-\pi K_\rho} X\Omega\|^2$$

still to express ψ_ρ and to be useful in proving its invariance properties, cf. [21] (3.3). We shall show this is in fact the case. To this end we need to discuss certain aspects of the spatial theory of von Neumann algebras, that may be of independent interest, and we collect them in Appendix A. We refer to this appendix for notations and notions used here below.

Notice now that equation (5.1) gives the dilation–translation commutation relations

$$\Delta(\psi_\rho/\omega')^{it} T_\rho(a) \Delta(\psi_\rho/\omega')^{-it} = T_\rho(e^{-2\pi t} a), \ a, t \in \mathbb{R},$$

so we may still apply the analysis made in Section 2.

We use the following notations:

$$\psi_a := \psi_\rho \circ \mathrm{Ad}(T_\rho(a)), \quad a > 0,$$

is a faithful normal weight on M;

$$\mathfrak{N} := \mathfrak{N}_{\psi_\rho}, \quad \mathfrak{N}_a := \{X \in M \ \mid \ \psi_a(X^*X) < \infty\},$$

are left ideals in M for any $a > 0$.

Note that $\sigma_t^{\psi_\rho}(\mathfrak{N}_a) = \mathfrak{N}_{e^{2\pi t}a}$, $a > 0$, $t \in \mathbb{R}$ and that if $X \in M$, and $a > 0$ then, cf. Prop. 4.2 and Lemma A.1,

$$X^*\Omega \in \mathcal{D}(\Delta(\psi_\rho/\omega')^{\frac{1}{2}} T_\rho(a)) \Leftrightarrow X \in \mathfrak{N}_a.$$

As a first result we show that the spectrum of the generator H_ρ of the translation group relative to the sector ρ always contains the positive real line.

5.1 Proposition. *Let $\mathcal{A}$ be a local net of von Neumann algebras on $\mathbb{R}$ as in Sect. 3 and ρ a covariant morphism localized in $(1, +\infty)$. Then $[0, +\infty) \subset Sp(H_\rho)$.*

Proof. We already know that $\mathfrak{N}$ is dense in M by the semifiniteness of ψ_ρ. By dilation covariance $\mathrm{Sp}(H_\rho)$ is either $\mathbb{R}_+ \cup \{0\}$, $\mathbb{R}_- \cup \{0\}$, or $\mathbb{R}$ thus we assume it to be $\mathbb{R}_- \cup \{0\}$ to find a contradiction. We then have $\mathcal{D}(\Delta(\psi_\rho/\omega')^{\frac{1}{2}}) \subset \mathcal{D}(\Delta(\psi_\rho/\omega')^{\frac{1}{2}} T_\rho(a)^*)$, $a > 0$, see Sect. 2. We take $X \in \mathfrak{N}^*$ and define $\zeta = T_\rho(a)^* X T(a)\Omega$, thus we have $\zeta \in \mathcal{D}(\Delta(\psi_\rho/\omega')^{\frac{1}{2}})$. Now we observe that the closed operator $T_\zeta \ \eta \ M$ as defined in Lemma A.3 coincides with the bounded operator $T_\rho(a)^* X T(a)$ which a priori is in M_{-a}; in fact the two operators coincide when restricted to the dense vector space $M'_{-a}\Omega$. Therefore T_ζ is bounded and $T_\zeta = T_\rho(a)^* X T(a) \in M$. It follows that $X \in M_a$ whenever $a > 0$ is small enough. Therefore we have $\mathfrak{N} \subset M_a$ thus $M \subset M_a$ which is not possible. $\square$

5.2 Lemma. $0 < b < a \Rightarrow \mathfrak{N} \cap \mathfrak{N}_a \subseteq \mathfrak{N} \cap \mathfrak{N}_b$.

Proof. By Lemma A.1 if $X^* \in \mathfrak{N} \cap \mathfrak{N}_a$, then $X\Omega \in \mathcal{D}(\Delta(\psi/\omega)^{\frac{1}{2}}) \cap \mathcal{D}(\Delta(\psi/\omega)^{\frac{1}{2}} T_\rho(a))$, thus by Corollary 2.3, $X\Omega \in \mathcal{D}(\Delta(\psi/\omega)^{\frac{1}{2}}) \cap \mathcal{D}(\Delta(\psi/\omega)^{\frac{1}{2}} T_\rho(b))$, hence $X^* \in \mathfrak{N} \cap \mathfrak{N}_b$ by Lemma A.1. □

5.3 Proposition. $\mathfrak{N} \subset \mathfrak{N}_a$ *for any* $a > 0$ *if and only if* $\mathfrak{N} = \cup_{a>0} \mathfrak{N} \cap \mathfrak{N}_a$.

Proof. One implication is obvious. To show the converse assume $\mathfrak{N} = \cup_{a>0} \mathfrak{N} \cap \mathfrak{N}_a$ and let $X \in \mathfrak{N}$ be fixed. Then there exists $a = a_1 > 0$ such that $X \in \mathfrak{N} \cap \mathfrak{N}_a$ i.e. $\psi_\rho \circ (X^*X) = \psi_\rho \circ \beta_a^\rho(X^*X) < \infty$, thus $\beta_a^\rho(X) \in \mathfrak{N}$; by iteration we find an increasing sequence $a_n > 0$ such that $\psi_\rho \circ \beta_{a_n}^\rho(X^*X) < \infty$.

Let us consider $a_* := \sup\{a \in \mathbb{R}_+ \mid X \in \mathfrak{N}_a\}$, $a_n \nearrow a_*$. But $\psi_\rho \circ \beta_{a_*}^\rho(X^*X) \leq \underline{\lim}_n \psi_\rho \circ \beta_{a_n}^\rho(X^*X) = \psi_\rho(X^*X)$ by lower semicontinuity, therefore $a_* = \infty$ and we are done. □

5.4 Proposition. *We have* $\cap_{a>0} \mathfrak{N}_a \subseteq \mathfrak{N}$.

Proof. If $X \in \cap_{a>0} \mathfrak{N}_a$, we have $\psi_\rho \circ \beta_a^\rho(X^*X) = \psi_\rho \circ \beta_b^\rho(X^*X)$ for every $a, b > 0$. □

We summarize the last results in the following proposition, although only the equivalence between the first two points will be needed.

5.5 Proposition. *The following assertions are equivalent:*

1) ρ *has positive energy;*

2) $\mathfrak{N} \subset \mathfrak{N}_a$ *for some (*$\Leftrightarrow$ *for all)* $a > 0$;

3) $\mathfrak{N} = \cup_{a>0}(\mathfrak{N} \cap \mathfrak{N}_a)$;

4) $\mathfrak{N} = \cap_{a>0} \mathfrak{N}_a$.

Proof. We shall show the equivalence 1) $\Leftrightarrow$ 2), the other results are immediate from Lemmata 5.2, 5.3, 5.4.
1) $\Rightarrow$ 2) From Lemma A.1 we have: $\mathfrak{N}^* = \{X \in M \mid X\Omega \in \mathcal{D}(\Delta(\psi/\omega')^{\frac{1}{2}})\}$ and $\mathfrak{N}_a^* = \{X \in M \mid X\Omega \in \mathcal{D}(\Delta(\psi/\omega')^{\frac{1}{2}} T_\rho(a))\}$, cf. Prop. 4.2. Now, as already recalled in the proof of Lemma 2.2 (from [21], Corollary 2.8), if ρ has positive energy, $\mathcal{D}(\Delta(\psi/\omega')^{\frac{1}{2}}) \subset \mathcal{D}(\Delta(\psi/\omega')^{\frac{1}{2}} T_\rho(a))$ for any $a > 0$ and so: $\mathfrak{N}^*\Omega \subset \mathfrak{N}_a^*\Omega$.
2) $\Rightarrow$ 1) From [25], p. 94–95, we have that $\mathfrak{N}^*\Omega$ is a core for $\Delta(\psi/\omega')^{\frac{1}{2}}$. If $\mathfrak{N}^* \subset \mathfrak{N}_a^*$, by Lemma A.1 $\mathfrak{N}^*\Omega \subset \mathcal{D}(\Delta(\psi/\omega')^{\frac{1}{2}} T_\rho(a))$ and so by Proposition 2.4 b), c), we get the positivity of the energy. □

5.2 Sectors of Haag dual nets on $\mathbb{R}$

In the following $\mathcal{A}$ is a net of von Neumann algebras on $\mathbb{R}$ satisfying the six properties listed in Sect. 3. Furthermore we require Haag duality on $\mathbb{R}$. Equivalently we assume our net to be strongly additive, meaning that $\mathcal{A}(a,b) \vee \mathcal{A}(b,c) = \mathcal{A}(a,c)$ for every $a < b < c$, $a, b, c \in \mathbb{R}$. It follows that $\mathcal{A}(a,b)' \cap \mathcal{A}(a,c) = \mathcal{A}(b,c)$ [19]. Let ρ be a covariant morphism of $\mathcal{A}$ localized in $(b, +\infty)$, $0 < b$,. As always

we assume that ρ is localizable in every half-line, thus it extends to a normal endomorphism of $\mathcal{A}(b,\infty)$. As recalled in Section 3, we obtain a localized cocycle $t \to z_\rho(-2\pi t) \in M := \mathcal{A}(0,\infty)$, thus a s.n.f. weight ψ_ρ on M such that $(D\psi_\rho : D\omega)_t = z_\rho(-2\pi t), t \in \mathbb{R}$. We have $U_\rho(-2\pi t) = \Delta(\psi_\rho/\omega')^{it}$ (see formula (5.1)).

For each $X \in M$ we have

$$\begin{aligned}\sigma_t^{\psi_\rho}(\rho(X)) &= z_\rho(-2\pi t)\alpha_{-2\pi t}(\rho(X))z_\rho^*(-2\pi t)\\ &= z_\rho(-2\pi t)(\alpha_{-2\pi t}\rho\alpha_{2\pi t}(\alpha_{-2\pi t}(X)))z_\rho^*(-2\pi t) = \rho(\alpha_{-2\pi t}(X))\\ &= \rho\alpha_{-2\pi t}\rho^{-1}(\rho(X)) = \sigma_t^{\omega\rho^{-1}}(\rho(X)),\ t \in \mathbb{R},\end{aligned}$$

therefore by Haagerup's theorem [25], p. 164, there exists a unique s.n.f. operator valued weight $E := E_\rho : M^+ \to \overline{\rho(M)}^+$ (where the symbol $\overline{\rho(M)}^+$ denotes the extended positive part of the von Neumann algebra $\rho(M)$, see Appendix A) such that $\psi_\rho = \omega\rho^{-1}E$. Here $\omega\rho^{-1}$ has to be thought of as the unique extension to $\overline{\rho(M)}^+$ such that $\omega\rho^{-1}(n) = n(\omega\rho^{-1})$, $n \in \overline{\rho(M)}^+$. In particular $E(X^*YX) = X^*E(Y)X$, for every $X \in \rho(M), Y \in M^+$. E may be uniquely "extended" to a (densely defined) linear mapping (also denoted by E) $\mathcal{M}_E := \text{lin}\{X \in M^+; E(X) \in \rho(M)^+\} \to \rho(M)$ with image a ultraweakly dense two–sided ideal in $\rho(M)$, such that $E(\rho(Y)X\rho(Z)) = \rho(Y)E(X)\rho(Z) \in \rho(M)$ for every $Y, Z \in M$, $X \in \mathcal{M}_E$. Clearly $\mathcal{M}_E \subseteq \mathcal{M}_{\psi_\rho}$. We consider the unbounded left inverse of ρ defined by $\phi_\rho := \rho^{-1} \circ E : M^+ \to \rho^{-1}(\overline{\rho(M)}^+) = \overline{M}^+$; by linearity $\phi_\rho : \mathcal{M}_E \to M$. Clearly $\rho\phi_\rho = E$, $\psi_\rho = \omega\phi_\rho$ both on M^+, $\mathcal{M}_E$, and $\phi_\rho(\rho(Y)X\rho(Z)) = Y\phi_\rho(X)Z$ for every $Y, Z \in M$, $X \in \mathcal{M}_E$.

5.6 Lemma. *With the notations above we have $E(X) \in \overline{M_a}^+$ for every $X \in {M_a}^+$ with $0 < a$ sufficiently small.*

Proof. Let $N := M_a' \cap M$, making use of the strong additivity property and duality we obtain that $N = \mathcal{A}(0,a)$ (it is immediate to see that $\mathcal{A}(0,a) \subset N$, on the other side if $x \in N$ then $x \in \mathcal{A}(0,a) = \mathcal{A}(0,a)'' = (M' \vee M_a)'$). If a is sufficiently small, so that ρ is localized in (a,∞) then $\rho(u) = u$ for every $u \in N$. For every $X \in M_a^+$ and unitary $u \in N$ we have $X = uXu^*$, so that $E(X) = E(uXu^*)$. Now using the fact that $u = \rho(u)$ (because $u \in N$), and that E is an operator valued weight, it follows that $E(X) = uE(X)u^*$ (in fact $E(uXu^*) = E(\rho(u)X\rho(u)^*) = \rho(u)E(X)\rho(u)^* = uE(X)u^*$). We know that $E(X)$ can be uniquely written as $he\ +\infty(I-e)$, with $e \in P(\rho(M)) \subset P(M)$ ($P(M)$ is the set of all the projections of M) and $h\ \eta\ e\rho(M)e$ positive. On the other side $E(X) = uE(X)u^* = uhu^*ueu^*\ +\infty(I - ueu^*)$ and using the uniqueness of the decomposition we obtain: $e = ueu^*$ and $h = uhu^*$ for every $u \in N$. From this, with the help of strong additivity ($N' \cap M = M_a$), we obtain $e \in M \cap N' = M_a$ (because $e \in P(M) \subset M$ and e commutes with every $u \in N$). In the same way we can say that $h\ \eta\ eM_ae$: the bounded parts of h are in eM_ae, but the bounded parts h_n of h are in N' (because h commutes with every unitary in N) and in eMe (because $h\ \eta\ eMe$); using the

fact that $eMe \cap N' = eM_a e$ (clearly $eM_a e \subset eMe \cap N'$ since $e \in M_a \subset N'$, and if $x \in eMe \cap N'$, $x = eme = e(eme)e$ with $eme \in M \cap N' = M_a$) we get $h_n \in eM_a e$ and thus $h \; \eta \; eM_a e$. From the fact that $E(X) = he + \infty(I - e)$ with $h \; \eta \; eM_a e$ and $e \in M_a$ it follows the result: $E(X) \in \overline{M_a}^+$. □

5.7 Lemma. *Let M, N be von Neumann algebras on the Hilbert space $\mathcal{H}$. We have $\overline{M}^+ \cap \overline{N}^+ = \overline{M \cap N}^+$.*

Proof. Both $\overline{M}^+$, $\overline{N}^+$ can be embedded in $\overline{B(\mathcal{H})}^+$. $m \in \overline{M}^+ \cap \overline{N}^+$ can be uniquely written as $he + \infty(I - e)$, with $e \in P(M)$, $h \; \eta \; eMe$, and $h'e' + \infty(I - e')$, with $e' \in P(N)$, $h' \; \eta \; e'Ne'$. It follows that $e = e' \in P(M \cap N)$, and $h = h' \; \eta \; M \cap N$. □

5.8 Lemma. *Let $M = \mathcal{A}(0, \infty)$, and ρ localized in $(b, +\infty)$, $0 < b$. Then we have*

$$\beta_a^\rho(M) \cap \rho(M) = \rho(\beta_a^\rho(M))$$

whenever $0 < a < b$.

Proof. The inclusion $\beta_a^\rho(M) \cap \rho(M) \supset \rho(\beta_a^\rho(M))$ is obvious. We choose $u_n \in (\rho, \rho_n)$ unitaries with ρ_n localized in $(-\infty, c_n)$ with $c_n \to -\infty$. Any weak limit point of the sequence $\mathrm{Ad}(u_n)$ is a map $\tilde{\phi} : B(\mathcal{H}) \to B(\mathcal{H})$ such that $\tilde{\phi}\rho = \mathrm{id}$ on $\cup_{l \in \mathbb{R}} \mathcal{A}(l, \infty)$. $\tilde{\phi}$ is a non normal left–inverse of ρ, cf. [11]. Let $X \in \beta_a^\rho(M) \cap \rho(M)$. Then $X = \rho(Y)$ for some $Y \in M$, and $Y = \tilde{\phi}(X)$. For every $Z \in \mathcal{A}(I)$, $I \subset (-\infty, a)$ we have $ZY = Z\tilde{\phi}(X) = \tilde{\phi}(\rho(Z)X) = \tilde{\phi}(ZX) = \tilde{\phi}(XZ) = YZ$ therefore Y commutes with $\cup_{I \subset (-\infty, a)} \mathcal{A}(I)$. By duality $Y \in \beta_a^\rho(M) = \alpha_a(M)$ and we are done. □

5.9 Proposition. *Let ρ be a covariant (transportable) morphism of $\mathcal{A}$ localized in $(b, +\infty)$, $b > 0$, $M := \mathcal{A}(0, \infty)$, $E : M^+ \to \overline{\rho(M)}^+$ defined as above, and $E_a := \rho\alpha_{-a}\rho^{-1}E\beta_a^\rho$. Then $E_a : M^+ \to \overline{\rho(M)}^+$ is a s.n.f. operator valued weight, for $0 < a < b$. If ρ is irreducible and ψ_a is semifinite, we have $E = E_a$, thus $\psi = \psi_a$.*

Proof. $X \in M^+ \Rightarrow \beta_a^\rho(X) \in {M_a}^+ \Rightarrow$

$$\begin{aligned} E\beta_a^\rho(X) &\in \overline{\rho(M)}^+ \cap \overline{M_a}^+ && \text{(by Lemma 5.6)} \\ &= \overline{\rho(M) \cap M_a}^+ && \text{(by Lemma 5.7)} \\ &= \overline{\rho(M_a)}^+ && \text{(by Lemma 5.8)} \end{aligned}$$

$\Rightarrow \rho^{-1}E\beta_a^\rho(X) \in \overline{M_a}^+ \Rightarrow \alpha_{-a}\rho^{-1}E\beta_a^\rho(X) \in \overline{M}^+ \Rightarrow \rho\alpha_{-a}\rho^{-1}E\beta_a^\rho(X) \in \overline{\rho(M)}^+$.

E_a is an operator valued weight. We check that for every $X, Y \in M$

$$\begin{aligned} E_a(\rho(Y)^* X \rho(Y)) &= \rho\alpha_{-a}\rho^{-1}E(\rho(\alpha_a(Y^*))\beta_a^\rho(X)\rho(\alpha_a(Y)) \\ &= \rho\alpha_{-a}(\alpha_a(Y^*)\rho^{-1}(E(\beta_a^\rho(X)))\alpha_a(Y)) \\ &= \rho(Y^*\alpha_{-a}\rho^{-1}(E(\beta_a^\rho(X)))Y) \\ &= \rho(Y^*)\rho(\alpha_{-a}(\rho^{-1}(E(\beta_a^\rho(X)))))\rho(Y) \\ &= \rho(Y^*)E_a(X)\rho(Y). \end{aligned}$$

E_a is normal, thus also faithful and semifinite (see [25], p. 155) since $\omega\rho^{-1}\circ E_a = \omega\rho^{-1}E\beta_a^\rho = \psi_\rho\beta_a^\rho$, which is faithful (since ψ_ρ is) and semifinite by hypothesis.

In the irreducible case it follows by uniqueness that $E = \lambda_a E_a$, $\lambda_a \in \mathbb{R}_+$, see e.g. [25], p.174 Corollary 12.13 (see also [12], Prop. 11.1), thus $\psi_\rho = \lambda_a\psi_a$ and $\lambda_a = 1$ by Sect. 2, 5. □

5.10 Corollary. *Let ρ be a covariant irreducible morphism of $\mathcal{A}$ localized in $(b, +\infty)$, $b > 0$. Then the following are equivalent:*

1) ψ_a is semifinite for some ($\Leftrightarrow$ for all) $a < b$;

2) ρ has positive energy;

3) $\psi_\rho = \psi_a$ for some ($\Leftrightarrow$ for all) $a > 0$.

Proof. It is immediate from Proposition 2.4, cf. Proposition 5.5, and Proposition 5.9. □

Our result entails the following one for sectors on non strongly additive nets.

5.11 Corollary. *Let $\mathcal{A}$ be a net on $\mathbb{R}$ satisfying the property of Section 3 (but not necessarily strongly additive). Let ρ be an irreducible covariant morphism of $\mathcal{A}$ localized in an bounded interval I. If ρ acts identically on $\mathcal{A}(1,\infty)'\cap\mathcal{A}(0,\infty)$, then ρ has positive energy iff ψ_a is semifinite.*

Proof. We may assume $I = (0, 1)$. ρ extends to a morphism $\tilde{\rho}$ of the dual net $\mathcal{A}^d$, covariant with respect to the same representation of the translation-dilation group. Since $\mathcal{A}^d(a, 0) = \mathcal{A}(0,\infty)' \cap \mathcal{A}(a,\infty)$, $a < 0$, and $\mathcal{A}^d(1, b)) = \mathcal{A}(b,\infty)' \cap \mathcal{A}(1,\infty)$, $b > 1$, it follows that $\tilde{\rho}$ is still localized in $(0, 1)$. Now $\mathcal{A}^d$ is strongly additive [19], thus $\tilde{\rho}$, hence ρ, has positive energy by Prop. 5.9. □

If (ρ, V_ρ) and (σ, V_σ) are two covariant morphisms, then $\rho\sigma$ is covariant with respect to the representation of P given by

$$V_{\rho\sigma}(g) := \rho(z_\sigma(g))V_\rho(g), g \in P \tag{5.2}$$

as shown by the following computation ($X \in M, g \in P$):

$$\begin{aligned} V_{\rho\sigma}(g)(\rho\circ\sigma(X))V_{\rho\sigma}(g)^* &= \rho\circ\sigma(V(g)XV(g)^*) = \rho(V_\sigma(g)\sigma(X)V_\sigma(g)^*) \\ &= \rho(z_\sigma(g)V(g)\sigma(X)V(g)^*z_\sigma(g)^*) \\ &= \rho(z_\sigma(g))\rho(V(g)\sigma(X)V(g)^*)\rho(z_\sigma(g)^*) \\ &= \rho(z_\sigma(g))V_\rho(g)\rho\circ\sigma(X)V_\rho(g)^*\rho(z_\sigma(g))^*. \end{aligned}$$

In the following we will show that if two irreducible sectors ρ, σ are covariant with positive energy with respect to the representations V_ρ, V_σ, then $V_{\rho\sigma}$ defined above is a positive energy representation.

5.12 Proposition. *Let $\mathcal{A}$ be a strongly additive net on $\mathbb{R}$, and ρ, σ be two covariant irreducible morphisms with positive energy. Then the representation $V_{\rho\sigma}$ defined by equation (5.2) has positive energy.*

Proof. To prove this we show the translational invariance of the unbounded left inverse $\phi_{\rho\sigma}$ (cf. the last subsection), namely $\alpha_a^{-1}\phi_{\rho\sigma}\beta_a^{\rho\sigma} = \phi_{\rho\sigma},\ a > 0$. As preliminary result we shall now prove that:

$$\phi_{\rho\sigma} = \phi_\sigma\phi_\rho \tag{5.3}$$

Let ψ be a s.n.f. weight on the von Neumann algebra M and ρ be an isomorphism of M onto its subalgebra $\rho(M)$. Then we have

$$\sigma_t^{\psi\rho^{-1}}(Y) = \rho \circ \sigma_t^\psi \circ \rho^{-1}(Y),\ Y \in \rho(M)$$

checking the invariance and the KMS property. From this, given two s.n.f. weights $\psi_i,\ i = 1, 2$, on M, we obtain by direct computation

$$\rho((D\psi_1 : D\psi_2)_t) = (D\psi_1\rho^{-1} : D\psi_2\rho^{-1})_t. \tag{5.4}$$

Now we have

$$z_{\rho\sigma}(t) = (D\psi_\sigma\rho^{-1}E_\rho : D\omega)_t \tag{5.5}$$

as shown by the following computation:

$$\begin{aligned}
z_{\rho\sigma}(t) = \rho(z_\sigma(t))z_\rho(t) &= \rho((D\psi_\sigma : D\omega)_t)(D\psi_\rho : D\omega)_t \\
\text{(by equation (5.4))} \qquad &= (D\psi_\sigma\rho^{-1} : D\omega\rho^{-1})_t(D\psi_\rho : D\omega)_t \\
\text{(by [25] Theorem 11.9)} \qquad &= (D\psi_\sigma\rho^{-1}E_\rho : D\omega\rho^{-1}E_\rho)_t(D\psi_\rho : D\omega)_t \\
\text{(since } \psi_\rho = \omega\rho^{-1}E_\rho) \qquad &= (D\psi_\sigma\rho^{-1}E_\rho : D\psi_\rho)_t(D\psi_\rho : D\omega)_t \\
\text{(by [25] Corollary 3.5)} \qquad &= (D\psi_\sigma\rho^{-1}E_\rho : D\omega)_t
\end{aligned}$$

From Formula (5.5), by a Theorem of Connes (see [25] Corollary 3.6) we deduce:

$$\psi_\sigma\rho^{-1}E_\rho = \psi_{\rho\sigma},$$

and from a Theorem of Haagerup (see [25] Theorem 11.9):

$$E_{\rho\sigma} = \rho E_\sigma\rho^{-1}E_\rho$$

and finally, applying $(\rho\sigma)^{-1}$, to both sides we get equation (5.3). The invariance of $\phi_{\rho\sigma}$ is readily obtained using equation (5.3) ($X \in M^+$):

$$\begin{aligned}
\alpha_a^{-1}\circ\phi_{\rho\sigma} \circ \beta_a^{\rho\sigma}(X) &= \alpha_a^{-1} \circ \phi_\sigma \circ \phi_\rho \circ \beta_a^{\rho\sigma}(X) \\
&= \alpha_a^{-1}(\phi_\sigma \circ \phi_\rho \circ \mathrm{Ad}(V_{\rho\sigma}(a))(X)) \\
\text{by equation (5.2)} \qquad &= \alpha_a^{-1}(\phi_\sigma \circ \phi_\rho \circ \mathrm{Ad}(\rho(z_\sigma(a))V_\rho(a))(X)) \\
&= \alpha_a^{-1}(\phi_\sigma(z_\sigma(a)\phi_\rho(\beta_a^\rho(X))z_\sigma(a)^*)) \\
&= \alpha_a^{-1}(\phi_\sigma(z_\sigma(a)(\alpha_a \circ \phi_\rho(X))z_\sigma(a)^*)) \\
&= \alpha_a^{-1}(\phi_\sigma(\beta_a^\sigma \circ \phi_\rho(X))) = \phi_\sigma \circ \phi_\rho(X).
\end{aligned}$$

The conclusion follows using Prop. 5.5, by the invariance of $\psi_{\rho\sigma}$. □

As already mentioned, the net on $\mathbb{R}$ we are considering are obtained by a local conformal precosheaf by removing a point from the circle [19]. Therefore our results have a version for Möbius covariant sectors.

5.13 Theorem. *Let $\mathcal{A}$ be a strongly additive local conformal precosheaf on S^1. The class of Möbius covariant (resp. traslation covariant, with respect to a given ∞ point) sectors with positive energy is stable under composition, conjugation and direct integral decomposition.*

Proof. The stability of the covariance with positive energy under direct integral decomposition is shown explicitly in Lemma 5.14 of the next subsection.

Let thus assume that ρ, σ are covariant sectors with positive energy. Let $\rho = \int^{\oplus} \rho_\lambda \mathrm{d}\mu(\lambda)$ and $\sigma = \int^{\oplus} \sigma_{\lambda'} \mathrm{d}\nu(\lambda')$ be two direct integral decompositions into irreducible sectors. By the previous statement, the irreducible components of ρ (resp. σ) are μ (resp. ν) almost everywhere covariant with positive energy. Then:

$$\rho\sigma = \int^{\oplus} \rho_\lambda \sigma_{\lambda'} \mathrm{d}(\mu \times \nu)(\lambda, \lambda')$$

therefore by Proposition 5.12 $\rho_\lambda \sigma_{\lambda'}$ is covariant with positive energy almost everywhere, and the same is true for $\rho\sigma$ by Lemma 5.14.

It remains to show that if ρ is covariant with positive energy, the same is true for its conjugate $\bar{\rho} = j \circ \rho \circ j$, where $j = \mathrm{Ad}J$ and J is the modular conjugation of (M, Ω). But this immediately follows by setting $V_{\bar{\rho}}(g) = JV_\rho(rgr)J$, where r is the change of sign on $\mathbb{R}$, see [16]. □

5.3 An example of sector with infinite dimension and negative energy levels

We show now that there exist translation-dilation covariant sectors whose energy spectrum is the real line. Our example, concerning a non strongly additive net, will be a reducible sector, but enlightens nevertheless the structure of the involved objects and the limit of the arguments.

5.14 Lemma. *Let $\rho = \int^{\oplus} \rho_\lambda d\mu(\lambda)$ be a (non unique) direct integral decomposition of a sector ρ of a net of von Neumann algebras $\mathcal{A}$ on $\mathbb{R}$ as in section 3 (resp. on S^1). Then ρ is translation (resp. Möbius) covariant with positive energy iff ρ_λ is translation (resp. Möbius) covariant with positive energy μ–almost everywhere.*

Proof. Clearly if ρ_λ is translation (resp. Möbius) covariant with positive energy for μ-almost all λ, the same is true for ρ. Conversely if ρ is translation covariant with positive energy, there exists by Borchers theorem [1] a unitary one-parameter group $T_\rho \in \rho(\mathcal{A})''$, with positive generator, implementing the translations $\rho \circ \mathrm{Ad}T(\cdot)$. Therefore T_ρ has a decomposition $T_\rho = \int^{\oplus} T_\rho^{(\lambda)} \mathrm{d}\mu(\lambda)$, where $T_\rho^{(\lambda)}$ has positive generator for almost all λ, and implements the translations on ρ_λ. If moreover ρ is covariant with respect to (the universal covering of

the) Möbius group, we may repeat the argument with the translations associated with different intervals of S^1 and find implementations in the weak closure of $\rho_\lambda(\mathcal{A})$. In this way we construct, for almost all λ, a unitary representation V_λ of the universal covering of the Möbius group up to a cocycle in the center of $\rho_\lambda(\mathcal{A})'$. Since the cohomology of the universal covering of the Möbius group is trivial, we may replace $V_\lambda(g)$ with $z(g)V_\lambda$, where $z(g)$ in the center of $\rho(\mathcal{A})'$, and get a true unitary representation providing the Möbius covariance with positive energy of ρ_λ. □

5.15 Proposition. *Let $\mathcal{A}$ be an irreducible net of von Neumann algebras on $\mathbb{R}$, covariant under translations and dilations as in Sect. 3. Let γ be a morphism of $\mathcal{A}$ localized in the interval $I \subset \mathbb{R}$, and assume that $\gamma_a := \alpha_{-a}\gamma\alpha_a$ is disjoint from γ for every $a \neq 0$ (thus γ is not translationally covariant). Then $\rho := \int^{\oplus} \gamma_a da$ is a translationally covariant (reducible) endomorphism with infinite statistics whose energy spectrum is $\mathbb{R}$.*

Proof. ρ acts on vectors ξ in the Hilbert space $L^2(\mathbb{R}, \mathcal{H})$ (separable if $\mathcal{H}$ is separable) via $(\rho(X)\xi)(a) = \gamma_{-a}(X)\xi(a)$, $X \in \mathcal{A}$, $a \in \mathbb{R}$, and covariance under translations is implemented by the unitary one–parameter group $(T(b)\xi)(a) = \xi(b-a)$. However ρ has not positive energy since otherwise we would infer from Lemma 5.14 that (for almost every $a \in \mathbb{R}$) γ_a is covariant and this is not possible by hypothesis. □

To give an explicit example, we recall that although the free scalar massless field $\varphi = \varphi(t,x)$ in two dimensions does not exist, its derivative $j = \partial_0\varphi - \partial_1\varphi$ makes sense and depends on $t - x$, thus defines a one dimensional current j. Every $f \in \mathcal{S}(\mathbb{R})$ such that $\int_{\mathbb{R}} f(t)dt = 0$ determines a unitary operator $W(f) = e^{ij(f)}$ and the Weyl relations $W(f+g) = e^{i\int fg'dt}W(f)W(g)$ are satisfied. Let $\mathcal{A}$ be the (strongly additive) net on $\mathbb{R}$ defined by $\mathcal{A}(I) := \{W(f) \mid f \in \mathcal{S},\ \int_{\mathbb{R}} f dt = 0,\ \mathrm{supp}(f) \subset I\}$. As is known [9], this net has localized automorphisms α_q, with $q \in \mathcal{S}$ real valued with compact support, given by $\gamma_q(W(f)) = e^{2i\int qf}W(f)$ [9]. If $q = Q'$ with $Q \in \mathcal{S}$, that is $\int q(t)dt = 0$, then γ_q is inner, indeed $\gamma_q = \mathrm{Ad}(W(Q))$. The equivalence class of these automorphisms are labeled by the real numbers $q_0 := \int q(t)dt$.

As noticed in [19], it is possible to generalize this construction in order to obtain non-covariant automorphisms, thus sectors with the properties needed in Prop. 5.15. For the sake of completeness we briefly recall this construction.

Let $\mathcal{B} \subset \mathcal{A}$ the net generated by the derivative of j: $\mathcal{B}(I) := \{W(f) \mid f = F',\ F \in \mathcal{S},\ \int F dt = 0,\ \mathrm{supp}(F) \subset I\}$. Then $\mathcal{B}$ is a proper subnet of $\mathcal{A}$ which is not strongly additive, but $\mathcal{A}(I) = \mathcal{B}(I)$ if I is a half-line. If q is a smooth function on $\mathbb{R}$ such that q' has compact support, then γ_q makes sense as automorphism of $\mathcal{B}$ and its equivalence class is labeled by the charges $\int q'(t)dt$ and $\int tq'(t)dt$. As a consequence, if $q(+\infty) \neq q(-\infty)$, then γ_q is a transportable localized automorphism of $\mathcal{B}$ such that $\alpha_{-a}\gamma_q\alpha_a$ is disjoint from γ_q for each non trivial translation α_a.

5.4 Construction of the Möbius covariant unbounded left inverse

Let $\mathcal{A}$ be a local conformal precosheaf on S^1, namely a map

$$I \to \mathcal{A}(I)$$

that associates to each (proper) interval of S^1 a von Neumann algebra, satisfying isotony, locality, Möbius covariance with positive energy, uniqueness of the vacuum, see e.g. [18]. A morphism ρ localized in the interval I_0 is a map

$$I \to \rho_I$$

which associates to every interval I of S^1 a normal representation of $\mathcal{A}(I)$ on $\mathcal{H}$ such that

$$\rho_{\tilde{I}}|_{\mathcal{A}(I)} = \rho_I, \ I \subset \tilde{I}$$

and

$$\rho_{I_0'} = \text{id}.$$

By Haag duality $\rho_I \in \text{End}(\mathcal{A}(I))$ if $I \supset I_0$.

We now assume that $\mathcal{A}$ is strongly additive. Then ρ is automatically covariant with positive energy if $d(\rho) < \infty$. In the following $d(\rho)$ is infinite and we assume Möbius covariance with positive energy.

By an *unbounded left inverse* ϕ of ρ we shall mean a map $I \to \phi_I$ that associates with any interval $I \supset I_0$ a map $\phi_I : \ \mathcal{A}(I)^+ \to \overline{\mathcal{A}(I)}^+$ such that

$$\phi_I(\rho_I(u)X\rho_I(u^*)) = u\phi_I(X)u^*, \ u, X \in \mathcal{A}(I)$$

i.e. $\rho_I\phi_I$ is a $\rho(\mathcal{A}(I))$–valued weight on $\mathcal{A}(I)$, and

$$\phi_{\tilde{I}}|_{\mathcal{A}(I)} = \phi_I$$

if $I \subset \tilde{I}$ are intervals containing I_0.

5.16 Proposition. *Let $\mathcal{A}$ be strongly additive and ρ irreducible, covariant with positive energy and localized in I_0. There exists an unbounded left inverse ϕ of ρ, covariant with respect to the Möbius group, namely such that*

$$\alpha_g^{-1} \circ \phi_{gI} \circ \beta_g^\rho \ = \ \phi_I$$

whenever $I \cap gI \supset I_0, \ g \in P$.

Proof. Let $\psi_I = \psi_{\rho,I}$ be the s.n.f. weight on $\mathcal{A}(I)$, $I \supset I_0$, defined by $(D\psi_I : D\omega_I)_t = z_\rho(-2\pi t), \ t \in \mathbb{R}$. Then

$$\psi_{\tilde{I}}|_{\mathcal{A}(I)} = \psi_I, \ I \subset \tilde{I} \qquad (5.6).$$

To check this notice that (5.6) is true if I and $\tilde{I}$ have a common boundary point, as by cutting the circle we may assume $I = (1, \infty)$, $\tilde{I} = (0, \infty)$ and we

may apply the results so far obtained for the real line. Thus (5.6) is true in general as we may check it in two steps by considering an intermediate interval $I \subset I_1 \subset \tilde{I}$ such that both $I \subset I_1$ and $I_1 \subset \tilde{I}$ have a common boundary point.

Concerning the invariance, let $I_1, I_2 \subset \tilde{I}$ be intervals containing I_0 and $g \in \mathtt{PSL}(2,\mathbb{R})$ with $gI_1 = I_2$. If $X_2 \in \mathcal{A}(I_2)$ then $X_2 = \beta^\rho_g(X_1)$ with $X_1 \in \mathcal{A}(I_1)$.

In fact $z_\rho(g) \in \mathcal{A}(gI_1)$ since I_0 and gI_0 are both contained in gI_1. Then

$$\begin{aligned} \psi_{gI_1}(\beta^\rho_g(X_1)) &= \psi_{I_2}(X_2) = \psi_{\tilde{I}}(X_2) \\ &= \psi_{\tilde{I}}(\beta^\rho_g(X_1)) = \psi_{\tilde{I}}(X_1) \\ &= \psi_{I_1}(X_1) \end{aligned} \tag{5.7}$$

provided g is a translation or dilation of $\tilde{I}$.

It follows that (5.7) holds for all g such that $gI, I \supset I_0$ by a simple repetition of the arguments. The reason is as follows: given two such intervals I, gI we can always find sequences $I_1 = I, \dots, I_n = gI$ and $\{\tilde{I}_i\}_1^{n-1}$ (actually $n = 3$) such that $I_i \cap I_{i+1} \supset I_0$, $I_i \cup I_{i+1} \subset \tilde{I}_i$, $I_{i+1} = g_i I_i$ with g_i a translation of $\tilde{I}_i$. Namely, if one of the two intervals $I_1 = I, I_3 = gI$ is contained in the other, let us take as I_2 an intermediate interval containing the smallest of the two and one of the endpoints of the biggest. Otherwise take as I_2 the connected component of I_0 in $I \cap gI$. In both of the situations it is easy to see that I_2 is an interval included in one of the two intervals I_1, I_3, containing the other one and having the two endpoints in common with I_1 and I_3 respectively. It is now possible to apply the previous results in two successive steps to the pairs I_1, I_2 and I_2, I_3 taking as $\tilde{I}_1$ and $\tilde{I}_2$ the biggest intervals and considering the unique translation g_1 of $\tilde{I}_1$ such that $g_1 I_1 = I_2$ (resp. g_2 of $\tilde{I}_2$ such that $g_2 I_2 = I_3$) with fixed point the common extreme of I_1 and I_2 (resp. I_2 and I_3). Then $g = g_1 g_2 h$ where h is a dilation of I and we have (by (5.7)): $\psi_{g_1 g_2 h I}(\beta^\rho_{g_1}\beta^\rho_{g_2}\beta^\rho_h(X)) = \psi_I(X)$, $X \in \mathcal{A}(I)$. Let $E_I : \mathcal{A}(I)^+ \to \overline{\rho_I(\mathcal{A}(I))}^+$ the Haagerup's operator valued weight, then

$$\phi_I := \rho_I^{-1} E_I$$

is the desired unbounded left inverse. □

5.17 Corollary. *Let $\mathcal{A}$ be strongly additive and ρ irreducible, covariant and localized in I_0. Then ρ has positive energy if and only if there exists an unbounded left inverse ϕ of ρ, covariant with respect to the Möbius group.*

Proof. The existence of the Möbius covariant unbounded left inverse has been shown in the preceding Proposition 5.16. The reverse implication follows from the invariance of ψ as in Corollary 5.10. □

A Appendix. Araki relative modular operators and Connes spatial derivatives

We use the notation in [25], Ch. I, II. In particular given a von Neumann algebra $M \subset B(\mathcal{H})$, a semifinite normal faithful (s.n.f.) weight ψ on M, $\mathfrak{N}_\psi$ will denote the dense left ideal $\{X \in M \mid \psi(X^*X) < \infty\}$, $\mathcal{H}_\psi$ the GNS Hilbert space of ψ. $\mathfrak{N}_\psi$ is embedded as a dense linear subspace of $\mathcal{H}_\psi$, denoted by $X \to (X)_\psi$, and the GNS representation π_ψ is given by $\pi_\psi(X)(Y)_\psi = (XY)_\psi$, $X \in M,\ Y \in \mathfrak{N}_\psi$. $\mathcal{D}(\mathcal{H}, \psi)$ will denote the dense linear subspace of all $\zeta \in \mathcal{H}$ that are ψ-bounded, namely the linear operator $R^\psi_\zeta : \mathcal{H}_\psi \to \mathcal{H}$ defined by $(X)_\psi \to X\zeta,\ X \in \mathfrak{N}_\psi$ is bounded. If χ' is a s.n.f. weight on M', $\Delta(\psi/\chi')$ denotes the Connes spatial derivative.[1]

A.1 Lemma. *Let $M \subset B(\mathcal{H})$ be a von Neumann algebra, $\omega' = (\Omega, \cdot\Omega)$ a vector state on M' given by a cyclic and separating vector Ω, ψ a normal faithful semifinite weight on M. Then $\|\Delta(\psi/\omega')^{\frac{1}{2}} X\Omega\|^2 = \psi(XX^*),\ X \in M$.*

Proof. As ω' is the state on M' given by a vector Ω, one checks immediately that $M\Omega = \mathcal{D}(\mathcal{H}, \omega')$ and $R^{\omega'}_{X\Omega} = X$. The Lemma is thus obtained by taking $\zeta = X\Omega$ in the formula $\|\Delta(\psi/\omega')^{\frac{1}{2}}\zeta\|^2 = \psi(R^{\omega'}_\zeta (R^{\omega'}_\zeta)^*)$ for $\zeta \in \mathcal{D}(\mathcal{H}, \omega')$ that defines the spatial derivative [25] p. 95. □

By polarization we also deduce that

$$(\Delta(\psi/\omega')^{\frac{1}{2}} Y\Omega, \Delta(\psi/\omega')^{\frac{1}{2}} X\Omega) = \psi(XY^*),\ X, Y \in \mathfrak{N}^*_\psi.$$

We also need two facts. Recall (see [25] Chapters I, II) that if ψ and ω are s.n.f. weight on M then the Tomita operator $S_{\psi,\omega} : \mathcal{H}_\omega \to \mathcal{H}_\psi$ is the closure of the densely defined anti-linear operator $(X)_\omega \to (X^*)_\psi$, $X \in \mathfrak{N}_\omega \cap \mathfrak{N}^*_\psi$. The polar decomposition $S_{\psi,\omega} = J_{\psi,\omega}\Delta^{\frac{1}{2}}_{\psi,\omega}$, $\Delta^{\frac{1}{2}}_{\psi,\omega} = (S^*_{\psi,\omega} S_{\psi,\omega})^{\frac{1}{2}}$, defines the relative modular conjugation and the relative modular operator.

Let $V_{\psi,\omega} : \mathcal{H}_\omega \to \mathcal{H}_\psi$ be the uniquely determined unitary operator satisfying: $\pi_\psi(X) = V_{\psi,\omega}\pi_\omega(X)V^*_{\psi,\omega}$, preserving the natural cones, see [25], 3.16. We shall identify $\mathcal{H}_\omega$ and $\mathcal{H}_\psi$ via $V_{\psi,\omega}$, cf. [25], 3.17, therefore $V_{\psi,\omega} = 1$ and $J_{\psi,\omega} = J_{\omega,\omega}$, that we simply denote by J in the following. We will assume M to act standardly on $\mathcal{H}_\psi = \mathcal{H}_\omega$, thus we suppress the symbols π_ψ and π_ω, and shall consider the s.n.f. weight $\omega' = \omega(J \cdot J)$ on M'.

A.2 Lemma. *Let M, ψ, ω' as above; then*

$$\Delta(\psi/\omega')^{\frac{1}{2}} = \Delta^{\frac{1}{2}}_{\psi,\omega}.$$

Proof. Let us first assume that M is a factor. Then $\Delta(\psi/\omega')^{it}\Delta^{-it}_{\psi,\omega} \in M \cap M' = \mathbb{C}$, $t \in \mathbb{R}$ because both $\Delta(\psi/\omega')^{it}$ and $\Delta^{it}_{\psi,\omega}$ implement the same modular groups

[1] If A is a positive linear operator, we set $\|A\zeta\| = +\infty$ for all vectors $\zeta \notin \mathcal{D}(A)$.

of ω' on M' and of ψ on M. It follows that $\Delta(\psi/\omega') = \lambda\Delta_{\psi,\omega}$ for some $\lambda > 0$. Now, for every $X \in \mathfrak{N}_\omega \cap \mathfrak{N}_\psi^*$, we have (cf. the proof of Lemma A.1)

$$\begin{aligned} \|\Delta_{\psi,\omega}^{\frac{1}{2}}(X)_\omega\|^2 &= \|J_{\psi,\omega}\Delta_{\psi,\omega}^{\frac{1}{2}}(X)_\omega\|^2 = \|(X^*)_\psi\|^2 \\ &= \psi(XX^*) = \psi(R^{\omega'}_{(X)_\omega} R^{\omega'}_{(X)_\omega}{}^*) \\ &= \|\Delta(\psi/\omega')^{\frac{1}{2}}(X)_\omega\|^2. \end{aligned} \tag{A.1}$$

In fact from the relation

$$JYJ(X)_\omega = XJ(Y)_\omega,\ Y \in \mathfrak{N}_\omega, \tag{A.2}$$

see [25], p. 26, it follows that $(X)_\omega \in \mathcal{D}(\mathcal{H}_\omega, \omega')$ since

$$\begin{aligned} (JYJ(X)_\omega, JYJ(X)_\omega)_\omega &= (XJ(Y)_\omega, XJ(Y)_\omega)_\omega = (J(Y)_\omega, X^*XJ(Y)_\omega)_\omega \\ &\le \|X\|^2((Y)_\omega, (Y)_\omega)_\omega = \|X\|^2\omega(Y^*Y) \\ &= \|X\|^2\omega'((JYJ)^*(JYJ)); \end{aligned}$$

furthermore $R^{\omega'}_{(X)_\omega} = X$ as results from:

$$\begin{aligned} R^{\omega'}_{(X)_\omega}(JYJ)_{\omega'} &= JYJ(X)_\omega \\ \text{(by equation (A.2))} \quad &= XJ(Y)_\omega \\ &= X(JYJ)_{\omega'} \end{aligned}$$

where, as before, $Y \in \mathfrak{N}_\omega$ and we have identified $(JYJ)_{\omega'}$ and $J(Y)_\omega$. Thus by equation (A.1), we obtain $\lambda = 1$. The conclusion follows by direct integral decomposition. $\square$

If A is a positive operator on $\mathcal{H}$ affiliated to M we define

$$\psi(A) := \sup_n \psi(AE_A([0,n))),$$

where E_A is the projection–valued spectral measure of A.

A.3 Lemma. *Let $M \subset B(\mathcal{H})$, ψ, ω', Ω as above; then*

$$\mathcal{D}(\Delta(\psi/\omega')^{\frac{1}{2}}) = \{T\Omega \mid T \textit{ closed},\ T\ \eta\ M,\ \Omega \in \mathcal{D}(T),\ \psi(TT^*) < \infty\}.$$

Proof. $\subset$: given $\zeta \in \mathcal{D}(\Delta(\psi/\omega')^{\frac{1}{2}})$, consider the densely defined operator $T_\zeta^0 : X'\Omega \to X'\zeta$, $X' \in M'$. Then T_ζ^0 is closable since its adjoint is densely defined; indeed $J(Y)_\psi \in \mathcal{D}(T_\zeta^{0*})$, $Y \in \mathfrak{N}_\psi$, and $T_\zeta^{0*}J(Y)_\psi = JY\Delta_{\psi,\omega}{}^{\frac{1}{2}}\zeta$ since

$$\begin{aligned} (T_\zeta^0 JX\Omega, J(Y)_\psi) &= (JXJ\zeta, J(Y)_\psi) = (\zeta, JX^*JJ(Y)_\psi) \\ &= (\zeta, JX^*(Y)_\psi) = (\zeta, J(X^*Y)_\psi) \\ &= (\zeta, JS_{\psi,\omega}Y^*X\Omega) = (\zeta, \Delta_{\psi,\omega}^{\frac{1}{2}}Y^*X\Omega) \\ &= (\Delta_{\psi,\omega}^{\frac{1}{2}}\zeta, Y^*X\Omega) = ((Y\Delta_{\psi,\omega}^{\frac{1}{2}}\zeta, X\Omega) = (JX\Omega, JY\Delta_{\psi,\omega}^{\frac{1}{2}}\zeta). \end{aligned}$$

Furthermore T_ζ^0 commutes with unitaries in M', thus its closure T_ζ is affiliated to M. Let $T_\zeta = VH$ be the polar decomposition, and $e_n := E_H([0,n))$, so that $T_n := T_\zeta e_n \in M$, and $T_n\Omega \to T_\zeta\Omega = \zeta$. Finally for each $X \in \mathfrak{N}_\psi^*$ we have

$$(T_n\Omega, S_{\psi,\omega}^* JX\Omega) = (JX\Omega, e_n J\Delta_{\psi,\omega}{}^{\frac{1}{2}}\zeta).$$

This is shown, using the fact that

$$\begin{aligned} S_{\psi,\omega}^*(JX\Omega) &= JS_{\psi,\omega}X\Omega \\ &= JS_{\psi,\omega}JX\Omega = J(X^*)_\psi, \end{aligned}$$

by the following computation:

$$\begin{aligned} (T_\zeta e_n\Omega, S_{\psi,\omega}^* JX\Omega) &= (T_\zeta e_n\Omega, J(X^*)_\psi) \\ &= (T_\zeta e_n\Omega, J(X^*)_\psi) = (e_n\Omega, T_\zeta^* J(X^*)_\psi) \\ &= (e_n\Omega, JX^*\Delta_{\psi,\omega}^{\frac{1}{2}}\zeta) = (\Omega, e_n JX^*\Delta_{\psi,\omega}^{\frac{1}{2}}\zeta) \\ &= (Je_n JX^*\Delta_{\psi,\omega}^{\frac{1}{2}}\zeta, \Omega) = (Je_n J\Delta_{\psi,\omega}^{\frac{1}{2}}\zeta, X\Omega) \\ &= (JX\Omega, e_n J\Delta_{\psi,\omega}^{\frac{1}{2}}\zeta) = (JX\Omega, e_n J\Delta_{\psi,\omega}^{\frac{1}{2}}\zeta), \end{aligned}$$

therefore $T_n\Omega \in \mathcal{D}(\Delta(\psi/\omega')^{\frac{1}{2}})$ and

$$S_{\psi,\omega}T_\zeta e_n\Omega = e_n J\Delta_{\psi,\omega}^{\frac{1}{2}}\zeta.$$

Moreover

$$\psi(T_nT_n^*) = \|\Delta(\psi/\omega')^{\frac{1}{2}}T_n\Omega\|^2 \leq \|\Delta(\psi/\omega')^{\frac{1}{2}}\zeta\|^2.$$

Finally $\psi(T_nT_n^*) \nearrow \psi(T_\zeta T_\zeta^*) < \infty$.

$\supset$: let $T_n := e_nT \in M$, where e_n is defined using the spectral family of TT^*; then $T_nT_n^* = e_nTT^*$ is increasing sequence of (bounded) positive operators, $T_nT_n^* \leq TT^*$, $\psi(T_nT_n^*) = \|\Delta(\psi/\omega')^{\frac{1}{2}}T_n\Omega\|^2 \leq \psi(TT^*) < \infty$. Therefore $T_n\Omega = e_nT\Omega \to \zeta$, and $\Delta(\psi/\omega')^{\frac{1}{2}}T_n\Omega$ is convergent i.e. it is a Cauchy sequence since $\Delta(\psi/\omega')^{\frac{1}{2}}(T_n - T_m)\Omega = \psi((T_n - T_m)(T_n - T_m)^*) = \psi((e_n - e_m)TT^*) = \psi(e_nTT^*) - \psi(e_mTT^*) \to 0$ when $n, m \to \infty$. In particular it follows that $\zeta \in \mathcal{D}(\Delta(\psi/\omega')^{\frac{1}{2}})$. □

Before concluding this appendix, we recall the notion of the extended positive part of a von Neumann algebra, needed in Sect. 6.

Given a von Neumann algebra M, its extended positive part $\overline{M}^+$ is defined as the family of all additive, positively homogeneous and lower semicontinuous functions $m : M_*^+ \to [0, \infty]$ [25], 11.1. $\overline{M}^+$ has the following characterization (see [25], 11.3): any element $m \in \overline{M}^+$ can be uniquely represented as $m = he + (1-e)\infty$ where e is a projection in M and h is a positive self–adjoint operator h affiliated to eMe. Every normal weight on M has a canonical extension to $\overline{M}^+$ such that $\varphi(m) = m(\varphi)$, $\varphi \in M_*^+$ (see [25], 11.4).

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